The Generalized Euler Equation and the Bankruptcy-Sovereign Default Problem

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Preliminary and Incomplete

Motivation

- Models of debt with default – household, firm and sovereign debt – are workhorses in the quantitative literature.
- These models are often solved numerically without characterizing the equilibrium, its existence and uniqueness.
- Precise characterization of trade-offs the borrower faces necessary to provide clear intuition and computation of these models.

What We Do

1. We revisit the case with short-term debt and no commitment.
2. Characterization through a Generalized Euler Equation (GEE):
   - Euler Equation with derivatives of future actions.
   - No expression for price derivatives is needed.
3. We then characterize the equilibrium with long-term debt:
   - Markov optimality conditions, existence and uniqueness.
   - Differentiability of price and policy functions.

What We Find

For long-term debt:

- The GEE enables a complete characterization of interior solutions.
- Equilibrium features two main regions:
  - Both dilution and default risks.
  - Only dilution risk.
- Equilibrium features two types of behavior:
  - Borrower chooses to never exceeds the risky debt limit.
  - Borrower enters the risky borrowing region with positive probability.
- Limit of finite horizon equilibrium exists and is unique.

Literature

- Incomplete markets models with default:
  - Auclert and Rognlie (2016), Aguiar et al. (2019), **Clausen and Strub (2020)**, Chatterjee and Eyigungor (2012) and **Aguiar and Amador (2020)**.

  ⇒ Long-term debt + GEE + Markov equilibrium.

- Generalized Euler equation:
  - Krusell et al. (2002, 2010), Krusell and Smith (2003), Klein et al. (2008), Arellano and Ramanarayanan (2012), Niepelt (2014) and Arellano et al. (2019).

  ⇒ Differentiability of price and policy functions.

- Quantitative solution methods:
  - Hatchondo et al. (2010) and Arellano et al. (2016).

  ⇒ Algorithm based on GEE and auxiliary functions.

Outline

Environment

- Risk averse borrower: standard $u(c)$ and $\beta < \frac{1}{1+r} \equiv \bar{p}$.
- Endowment $y \in [\underline{y}, \bar{y}]$ is *iid* with continuous cdf F and density f.
- Borrowing of non-contingent debt in competitive lending market.
- Borrowing $b > 0$, debt pays coupon 1, fraction λ of the debt matures.
- Timing: (1) y realizes, (2) repay or not, (3) new bond issue or autarky.
- Upon default, borrower suffers permanent financial autarky:

$$V^A(y) = u(y) + \frac{\beta}{1-\beta} E[u(c)] = u(y) + \beta \bar{v}.$$

Outline

Short-Term Debt

- Discrete choice:

$$V(y,b) = \max\left\{ \overbrace{V^R(y,b)}^{\text{repayment}}, \overbrace{V^A(y)}^{\text{default}} \right\}.$$

- Value of repayment:

$$\begin{aligned} V^R(y,b) &= \max_{b'}\left\{ u\big(y - b + q(b')b'\big) + \beta \int_{\underline{y}}^{\bar{y}} V(y',b')dF \right\} \\ &= \max_{b'}\left\{ u\big(y - b + q(b')b'\big) + \underbrace{\beta \int_{d(b')}^{\bar{y}} \left\{ V^R(y',b') - V^A(y') \right\} dF}_{\text{value of access to credit markets}} + \beta\bar{v} \right\}. \end{aligned}$$

- Default threshold:

$$d(b) = \min\left\{ \{y : V^R(y,b) \geq V^A(y)\} \cup \{\bar{y}\} \right\}.$$

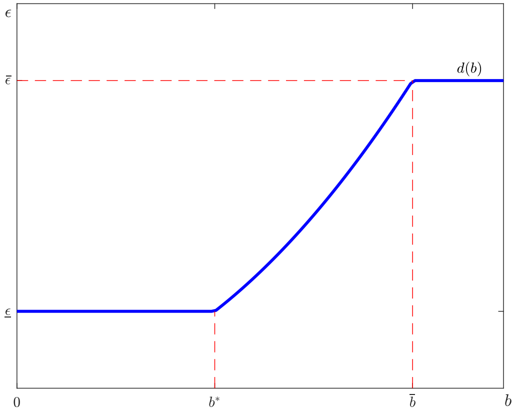
- Risk-free borrowing threshold: $b^* \geq 0$ such that $V^R(\underline{y}, b^*) = V^A(\underline{y})$.

Short-Term Debt: GEE

$$u_c(c)\underbrace{\left[q(b') + q_b(b')b'\right]}_{\text{marginal revenue}} = \beta \int_{d(b')}^{\bar{y}} u_c(c')dF$$

- Is this price differentiable? Almost, but not quite.

Default Threshold



- For debt $b > b^*$ there is default risk.
- $d(b)$ not differentiable at b^*. $\partial^+ d(b) > 0$, but $\partial^- d(b) = 0$.
- No analytical solution for b^*, but we know it solves $V^R(\underline{y}, b^*) = V^A(\underline{y})$.

Short-Term Debt: Bond Price

- Bond price:

$$q(b') = \begin{cases} \bar{p}[1 - F(d(b'))], & b^* < b' \\ \\ \bar{p}, & b' \leq b^* \end{cases}$$

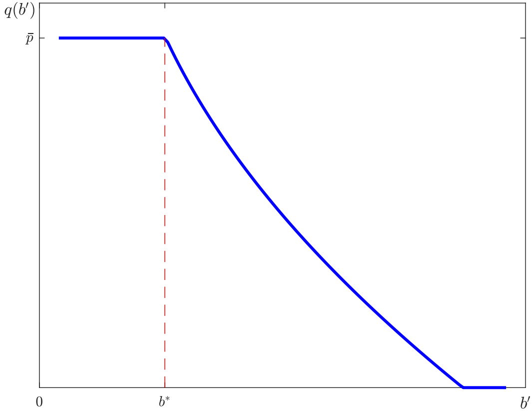
- Derivative is defined for $b' \neq b^*$ (inherited property of $d(b)$):

$$q_b(b') = -\bar{p} f(d(b')) d_b(b').$$

- Marginal revenue of borrowing at b':

$$q(b') + q_b(b')b' = \bar{p}[1 - F(d(b))] - \bar{p} f(d(b')) d_b(b') b'.$$

Short-Term Debt: Bond Price



- The kink in the price at the risk-free borrowing limit b^* makes b^* more attractive.
- Agents will choose to stay at b^* to avoid lowering the price of their debt.

Short-Term Debt: GEE

- From Clausen and Strub (2020) we know either:

  1. $b' = b^*$

  2. $b' > b^*$ and solves the GEE:

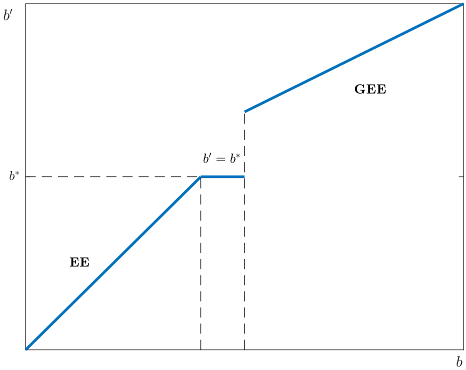
  $$u_c(c)[\big(1 - F(d(b'))\big) - f(d(b'))d_b(b')b'] = \beta R \int_{d(b')}^{\bar{y}} u_c(c')dF.$$

  3. $b' < b^*$ and solves EE:

  $$u_c(c) = \beta R \int u_c(c')dF$$

- No need to consider the price explicitly.

Short-Term Debt: Borrowing Policy



- Agents stay at the risk-free limit b^* to avoid lowering price of debt

Outline

Long-Term Debt: What's Different? Dilution

- Consumption with long maturity bonds:

$$c = y - b + q(b')[b' - (1-\lambda)b].$$

- Sovereign's choice of borrowing determines the value of outstanding debt $b[q(b')(1-\lambda)b - 1]$

- Since debts can be diluted by sovereign, price today depends on future actions. Sovereign cannot commit not to borrow more in the future.

- This is a harder problem to characterize without the price.

Long-Term Debt: Borrower's Problem

- The value of repayment:

$$V^R(y,b) = \max_{b'} \Big\{ u\big(y - b + q(b')\left[b' - (1-\lambda)b\right]\big) + \beta \int_{\underline{y}}^{\bar{y}} V(y',b')dF \Big\}.$$

- Bond policy function: $b' = h(y,b)$.

- What would a GEE look like (when it holds)?

$$u_c(c)\big[q(b') + q_b(b')[b' - (1-\lambda)b]\big] = \beta \int_{d(b')}^{\bar{y}} u_c(c')\big[1 + (1-\lambda)q(b'')\big]\,dF.$$

- Depends on price derivative $q_b(b')$ as in the case of short-term debt.

Long-Term Debt: Bond Price

- Bond price:

$$q(b') = \bar{p} \int_{\underline{y}}^{\bar{y}} \mathbb{I}_{\{V^R(y',b') \geq V^A(y')\}} \left[1 + (1-\lambda) q(h(y',b'))\right] dF$$
$$= \bar{p}\left[1 - F(d(b'))\right] + \bar{p}(1-\lambda) \int_{d(b')}^{\bar{y}} q\big(h(y',b')\big) dF$$

- Price depends on both default $d(b')$ and future borrowing $h(b', y')$.

- Changes in price due to $d(b')$ reflect default risk, those due to $h(y', b')$ reflect dilution risk.

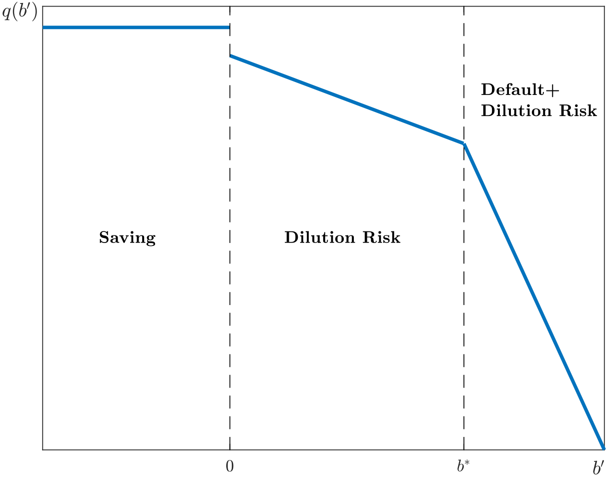
Long-Term Debt: Bond Price

- Bond Price

$$q(b') = \begin{cases} \bar{p}[1 - F(d(b'))] + \bar{p}(1-\lambda)\int_{d(b')}^{\bar{y}} q(h(y',b'))dF, & b^* < b' \\ \bar{p} + \bar{p}(1-\lambda)\int_{\underline{y}}^{\bar{y}} q(h(y',b'))dF, & 0 < b' \leq b^* \\ \frac{1}{r+\lambda}, & b' \leq 0 \end{cases}$$

- With short-term debt ($\lambda = 1$), $q(b') = \bar{p}$ when $b' < b^*$.
- With $\lambda < 1$, debt is honored next period with certainty, but dilution risk.
- Why? If there is probability of $b' > b^*$ at some point (after a sequence of bad shocks), the price today reflects this risk.

Long-Term Debt: Bond Price



- With long-term debt there is a discount for dilution risk at $b' = 0$.

Long-Term Debt: Bond Price

Derivative for $b' \notin \{0, b^*\}$

$$q_b(b') = \bar{p}(1-\lambda) \underbrace{\int_{d(b')}^{\bar{y}} q_b(h(\cdot))h_b(\cdot)dF}_{\text{Dilution, } b'>0} - \bar{p} \overbrace{\overbrace{\left[1 + (1-\lambda)q(h(d(b'), b'))\right]}^{\text{Value of loss}} \overbrace{f(d(b'))d_b(b')}^{\text{Marginal P(default)}}}^{\text{Default, } b'>b^*}$$

Leads to three cases for our GEE:

1. Borrowing $b' > b^*$ has both default and dilution terms
2. Borrowing $0 < b' < b^*$ has dilution risk only
3. Saving $b < 0$ has neither

Long-Term Debt: Bond Price

- Is this dilution term well-defined?

$$\int_{d(b')}^{\bar{y}} q_b(h(\cdot))h_b(\cdot)dF$$

- Yes, there are three types of points $y \in [d(b'), \bar{y}]$.

1. Points s.t. $b' \notin \{0, b^*\}$, and h_b, $q_b(h)$ are defined.
2. Points s.t. $b' \in \{0, b^*\}$, and $h_b = 0$, $\Rightarrow q_b(h)h_b = 0$.
3. The remaining points where $b' \in \{0, b^*\}$, and h_b, hence the integrand $q_b(h)h_b$, is not well-defined but has measure zero.

Long-Term Debt: Eliminating $q_b(b')$

- Use value of q_b implied by GEE, call it $B(h, d, q)$:

$$q_b = B(h, d', q) = \frac{\int_{d'} u_c[1 + (1-\lambda)q']dF - u_c(c)q}{u_c[h - (1-\lambda)b]}$$

- Substitute this into the expression for the bond price derivative:

$$q_b = \bar{p}(1-\lambda)\int_{d(b')}^{\bar{y}} B(h', d'', q')h_b dF - \bar{p}\left[1 + (1-\lambda)\tilde{q}\right] f(d)d_b$$

- Substitute back into GEE:

$$u_c(c)\left[q(b') + \overbrace{\left\{\bar{p}(1-\lambda)\int_{d(b')}^{\bar{y}} B(h', d'', q')h_b dF - \bar{p}\left[1 + (1-\lambda)\tilde{q}\right] f(d)d_b\right\}}^{q_b(b')}[b' - (1-\lambda)b]\right]$$
$$= \beta\int_{d(b')}^{\bar{y}} u_c(c')\left[1 + (1-\lambda)q(b'')\right] dF$$

Default Threshold

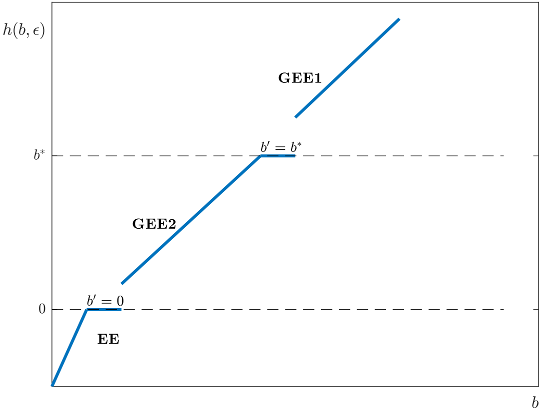
Long-Term Debt: GEE Effects

$$u_c(c)\Bigg[\overbrace{q(b')}^{\text{consumption gain from marginal borrowing}} + \underbrace{\left\{\bar{p}(1-\lambda)\int_{d(b')}^{\bar{y}} B(h',d'',q')h_b dF\right\}[b'-(1-\lambda)b]}_{\text{dilution},\ b'>0} - \underbrace{\left\{\bar{p}\left[1+(1-\lambda)\tilde{q}\right]f(d)d_b\right\}[b'-(1-\lambda)b]}_{\text{default},\ b'>b*}\Bigg] = \beta\int_{d(b')}^{\bar{y}} u_c(c')\left[1+(1-\lambda)q(b'')\right]dF$$

Two borrowing regions that reflect different risks to creditors:

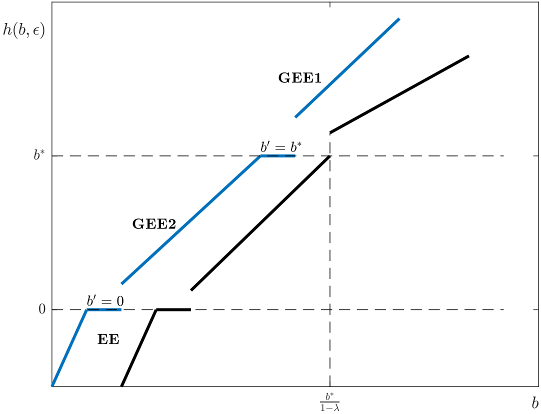
1. $b' > b^*$, the GEE reflects both default and dilution risk (GEE1)
2. $0 < b^{'} < b^*$, the GEE reflects only dilution risk (GEE2)

Borrowing Policy: When Dilution is Positive



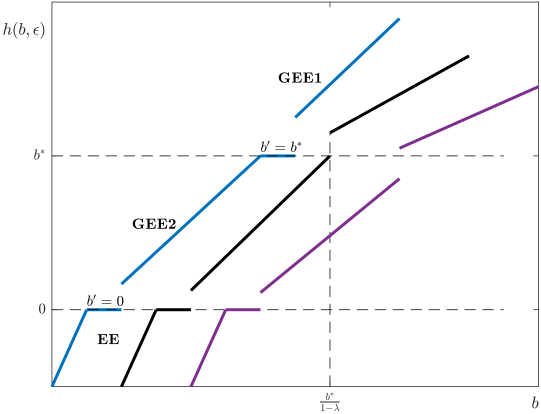
- Agents wait to borrow, due to dilution lowering the price of borrowing.
- As with short-term debt, agents stay at risky borrowing limit b^*.
- There is a discontinuity at b^*, due to the kink in the pricing.

Borrowing Policy: When Dilution is Zero



- Agents do not stay at b^* when no net borrowing.
- Discontinuities are smaller: less is borrowed because the price is higher.

Borrowing Policy: When Dilution is Negative



- There is a jump direct to higher debt without going through b^*.
- As with short-term debt, agents stay at b^* if dilution effect is large enough.

Long-Term Debt: Existence and Uniqueness

- Operator K(q) on price:

$$\big(Kq\big)(b') = \bar{p}\big[1 - F(d(b';q))\big] + \bar{p}(1-\lambda)\int_{d(b';q)}^{\bar{y}} q\big(h(y',b';q)\big)dF.$$

- Note that $d(\cdot;q)$ and $h(\cdot;q)$ being implicit functions of q.
- Chatterjee and Eyigungor (2012) show existence of a fixed point q^* that is decreasing in b'.
- Aguiar and Amador (2020) show potential multiplicity in q^*.

Long-Term Debt: Existence and Uniqueness

- We impose a restriction on $q(b')$ that it be the limit of a finite horizon model as $T \to \infty$.
- Specifically, we consider the price in the first period of a finite horizon model $q_1(b'; T)$ as T becomes large.
- We use backwards induction starting at $q_T(b'; T) = 0$ to get $q_{T-1}(b'; T), \ \ldots$, until $q_1(b'; T)$.
- We show the limit exists and is unique.
- Aguiar and Amador (2020) may not restrict to Markov equilibria.

Long-Term Debt: Differentiability

- Prices and debt functions exhibit jumps in various places.
- Those jumps usually prevent differentiability.
- We add extreme value shocks and consider only $b \geq 0$.
- Price and policy function are differentiable almost everywhere.
- For arbitrarily small scale parameter, derivation of the GEE is possible.

Long-Term Debt: Summary

We can describe equilibrium as set of functional equations in h and d

1. Auxiliary Functions

$$q(h(y,b)) = \bar{p}\left\{[1-F(d)] + (1-\lambda)\int_d q(h(h))dF\right\}$$

$$B(y,b;h,d,q) = \frac{\int_{d'} u_c[1+(1-\lambda)q']dF - u_c q}{u_c[h-(1-\lambda)b]}$$

$$V^R(y,b) = u(y - bq[h-(1-\lambda)b) + \int_d V^R - V^A dF + \beta\bar{v}$$

2. Equilibrium functional equations

$$u_c(c)\left[q(b') + \left\{\bar{p}(1-\lambda)\int_{d(b')}^{\bar{y}} B(h',d'',q')h_b dF - \bar{p}\left[1+(1-\lambda)\tilde{q}\right]f(d)d_b\right\}[b'-(1-\lambda)b]\right]$$
$$= \beta\int_{d(b')}^{\bar{y}} u_c(c')\left[1+(1-\lambda)q(b'')\right]dF$$

$$V^R(d,y) = V^A(d), \qquad V^R(\underline{y},b^*) = V^A(\underline{y})$$

Computation

- The most common way to solve these models is value function iteration on a discrete grid. Very slow. Need to iterate between $V(y, b; q)$ and q.
- Arellano et al. (2016) use Euler equation to solve short-term debt problem numerically, but assume the GEE always holds.
- Hatchondo et al. (2010) compare various VFI algorithms to solve the short-term debt problem, but assess their accuracy using Euler residuals.
- Our characterization suggests using a numerical approach based on the GEE and auxiliary equations.

Outline

Conclusion

- We characterize the equilibrium of unilateral default problem without commitment.
- We use the GEE both to gain insight into the nature of the equilibrium and as a basis for computations.
- If marginal revenue is well-defined, the GEE describes the optimal borrowing policy.
- The GEE fails to capture tradeoffs at choices where the price is not differentiable, but we can still describe the optimal policy.

Thanks for your attention!

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Long-Term Debt: Default Threshold

We can take a closer look at the derivative of the default threshold

$$d_b(b') = \frac{u_c\big(c(d(b'), b')\big)[1 + (1-\lambda)q(b'')]}{u_c\big(c(d(b'), b')) - u_c\big(d(b')\big)} > 1$$

- Numerator is marginal utility loss from additional debt after repayment.
- Denominator cost, in terms of marginal utility, to maintain access to financial markets.

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